- A complex function maps complex numbers to complex numbers. For example, the function $F(z) = z^2 + 2z + 1$, where z is complex, is a complex function.
- A complex polynomial function is a mapping of the form

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where $z, a_0, a_1, \ldots, a_n$ are complex.
- A complex rational function is a mapping of the form

$$F(z) = \left(\frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m} \right),$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$ and z are complex.
- Observe that a polynomial function is a special case of a rational function.
- Herein, we will mostly focus our attention on polynomial and rational functions.

- A function F is said to be continuous at a point z_0 if $F(z_0)$ is defined and given by

$$F(z_0) = \lim_{z \to z_0} F(z($$

- A function that is continuous at every point in its domain is said to be continuous.
- Polynomial functions are continuous everywhere.
- Rational functions are continuous everywhere except at points where the denominator polynomial becomes zero.

- A function F is said to be differentiable at a point $z = z_0$ if the limit

$$F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0}$$

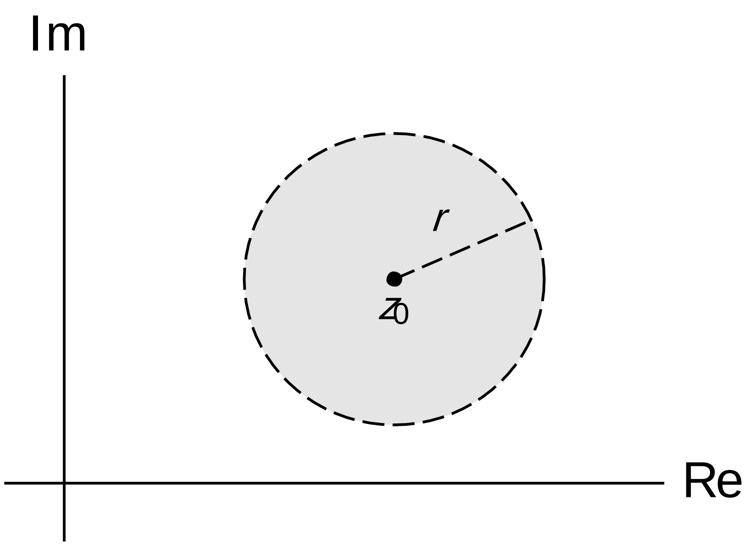
  exists. This limit is called the derivative of F at the point $z = z_0$.
- A function is said to be differentiable if it is differentiable at every point in its domain.
- The rules for differentiating sums, products, and quotients are the same for complex functions as for real functions. If $F'(z_0)$ and $G'(z_0)$ exist, then
  1. $(aF)'(z_0) = aF'(z_0)$ for any complex constant a;
  2. $(F + G)'(z_0) = F'(z_0) + G'(z_0)$;
  3. $(FG)'(z_0) = F'(z_0)G(z_0) + F(z_0)G'(z_0)$;
  4. $(F/G)'(z_0) = \frac{G(z_0)F'(z_0) - F(z_0)G'(z_0)}{G(z_0)^2}$; and
  5. if $z_0 = G(w_0)$ and $G'(w_0)$ exists, then the derivative of $F(G(z))$ at w_0 is $F'(z_0)G'(w_0)$ (i.e., the chain rule).
- A polynomial function is differentiable everywhere.
- A rational function is differentiable everywhere except at the points where its denominator polynomial becomes zero.

- An open disk in the complex plane with center z_0 and radius r is the set of complex numbers z satisfying

$$|z - z_0| < r,$$

where r is a strictly positive real number. A

- plot of an open disk is shown below.



- •A function is said to be analytic at a point z_0 if it is differentiable at every point in an open disk about z_0
- • A function is said to be analytic if it is analytic at every point in its domain. A polynomial function is analytic everywhere.
- •A rational function is analytic everywhere, except at the points where its denominator polynomial becomes zero.

- If a function F is zero at the point z_0 (i.e., $F(z_0) = 0$), F is said to have a zero at z_0
- If a function F is such that $F(z_0) = 0, F^{(1)}(z_0) = 0, \ldots, F^{(n-1)}(z_0) = 0$ (where $F^{(k)}$ denotes the kth order derivative of F), F is said to have an nth order zero at z_0
- A point at which a function fails to be analytic is called a singularity.
- Polynomials do not have singularities.
- Rational functions can have a type of singularity called a pole.
- If a function F is such that $G(z) = 1/F(z)$ has an nth order zero at z_0, F is said to have an nth order pole at z_0
- A pole of first order is said to be simple, whereas a pole of order two or greater is said to be repeated. A similar terminology can also be applied to zeros (i.e., simple zero and repeated zero.(

- Given a rational function F, we can always express F in factored form as

$$F(z) = \frac{K(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\cdots(z-a_M)^{\alpha_M}}{(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\cdots(z-b_N)^{\beta_N}},$$

  where K is complex, $a_1, a_2, \ldots, a_M, b_1, b_2, \ldots, b_N$ are distinct complex numbers, and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $\beta_1, \beta_2, \ldots, \beta_N$ are strictly positive integers.
- One can show that F has *poles* at $b_1, b_2, \ldots, b_N$ and *zeros* at $a_1, a_2, \ldots, a_M$.
- Furthermore, the kth pole (i.e., b_k) is of *order* β_k, and the kth zero (i.e., a_k) is of *order* α_k.
- When plotting zeros and poles in the complex plane, the symbols “o” and “x” are used to denote zeros and poles, respectively.

Part 13

Partial Fraction Expansions (PFEs)

- Sometimes it is beneficial to be able to express a rational function as a sum of *lower-order* rational functions.
- This can be accomplished using a type of decomposition known as a partial fraction expansion.
- Partial fraction expansions are often useful in the calculation of inverse Laplace transforms, inverse z transforms, and inverse CT/DT Fourier transforms.

- Consider a rational function

$$F(v) = \frac{\alpha_m v^m + \alpha_{m-1} v^{m-1} + \ldots + \alpha_1 v + \alpha_0}{\beta_n v^n + \beta_{n-1} v^{n-1} + \ldots + \beta_1 v + \beta_0}.$$

- The function F is said to be strictly proper if $m < n$ (i.e., the order of the numerator polynomial is strictly less than the order of the denominator polynomial.(
- Through polynomial long division, any rational function can be written as the sum of a polynomial and a strictly-proper rational function.
- A *strictly-proper* rational function can be expressed as a sum of lower-order rational functions, with such an expression being called a partial fraction expansion.

- Any rational function can be expressed in the form of

$$F(v) = \frac{a_m v^m + a_{m-1} v^{m-1} + \ldots + a_0}{v^n + b_{m-1} v^{m-1} + \ldots + b_0}.$$

- Furthermore, the denominator polynomial $D(v) = v^n + b_{m-1} v^{m-1} + \ldots + b_0$ in the above expression for $F(v)$ can be factored to obtain

$$D(v) = (v - p_1)^{q_1} (v - p_2)^{q_2} \cdots (v - p_n)^{q_n},$$

  where the p_k are distinct and the q_k are integers.
- If F has only simple poles, $q_1 = q_2 = \cdots = q_n = 1$.
- Suppose that F is strictly proper (i.e., $m < n$.(
- In the determination of a partial fraction expansion of F, there are *two cases* to consider:
  1. F has *only simple poles*; and
  2. F has *at least one repeated pole*.

- Suppose that the (rational) function F has only simple poles.
- Then, the denominator polynomial D for F is of the form

$$D(v) = (v - p_1)(v - p_2) \cdots (v - p_n),$$

  where the p_k are distinct.
- In this case, F has a partial fraction expansion of the form

$$F(v) = \frac{A_1}{v - p_1} + \frac{A_2}{v - p_2} + \ldots + \frac{A_{n-1}}{v - p_{n-1}} + \frac{A_n}{v - p_n},$$

  where

$$A_k = (v - p_k) F(v)|_{v = p_k}.$$

- Note that the (simple) pole p_k contributes a single term to the partial fraction expansion.

- Suppose that the (rational) function F has at least one repeated pole. One
- can show that, in this case, F has a partial fraction expansion of the form

$$F(v) = \left[\frac{A_{11}}{v-p_1} + \frac{A_{12}}{(v-p_1)^2} + \ldots + \frac{A_{1q_1}}{(v-p_1)^{q_1}}\right] + \left[\frac{A_{21}}{v-p_2} + \ldots + \frac{A_{2q_2}}{(v-p_2)^{q_2}}\right] + \ldots + \left[\frac{A_{P1}}{v-p_P} + \ldots + \frac{A_{Pq_P}}{(v-p_P)^{q_P}}\right]$$

where

$$A_{kl} = \frac{1}{(q_k-l)!}\left[\frac{d^{q_k-l}}{dv^{q_k-l}}\left[(v-p_k)^{q_k}F(v)\right]\right]\Bigg|_{v=p_k}.$$

- Note that the q_kth-order pole p_k contributes q_k terms to the partial fraction expansion.
- Note that $n! = (n)(n-1)(n-2)\cdots(1)$ and $0! = 1$.

Part 14

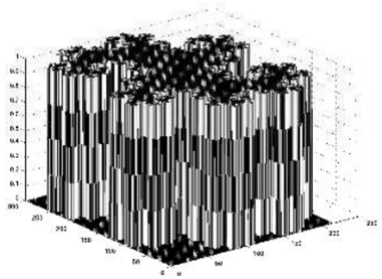
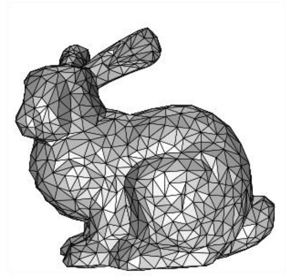
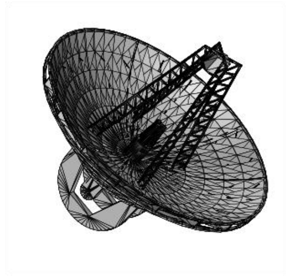
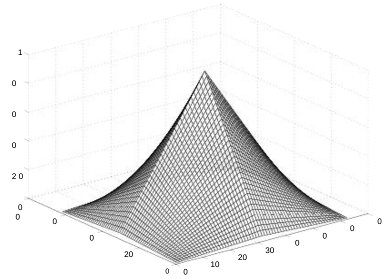
Epilogue

ELEC 486: Multiresolution Signal and Geometry Processing with C++

- If you did not suffer permanent emotional scarring as a result of using these lecture slides and you happen to be a student at the University of Victoria, you might consider taking the following course (developed by the author of these lecture slides) as one of your technical electives (in third or fourth year):

  ELEC 486: Multiresolution Signal and Geometry Processing with C++

- Some further information about ELEC 486 can be found *on the next slide*, including the URL of the course web site.

ELEC :486/586
Multiresolution Signal and Geometry Processing with C++

- normally offered in Summer (May-August) term; only prerequisite ELEC 310
- subdivision surfaces and subdivision wavelets
  - 3D computer graphics, animation, gaming (Toy Story, Blender software)
  - geometric modelling, visualization, computer-aided design
- multirate signal processing and wavelet systems
  - sampling rate conversion (audio processing, video transcoding)
  - signal compression (JPEG 2000, FBI fingerprint compression)
  - communication systems (transmultiplexers for CDMA, FDMA, TDMA(
- C++ (classes, templates, standard library), OpenGL, GLUT, CGAL
- software applications (using C(++
- for more information, visit course web page: http://www.ece.uvic.ca/~mdadams/courses/wavelets